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VIBRATIONS OF CIRCULAR TUBES AND SHELLS FILLED AND PARTIALLY IMMERSED IN DENSE FLUIDS

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Free vibrations of circular cylindrical shells and tubes completely filled with a dense fluid and partially immersed in a different fluid (liquid) having a free surface are studied. Elastic shell constraints, varying from simply supported to clamped ends are assumed. Fluids are assumed to be stationary, inviscid and incompressible. The fluid outside the shell is assumed to be unlimited in the radial direction and limited in the vertical direction by a rigid bottom and a free surface. The effect of free surface waves is considered, so that both sloshing and bulging modes of the system are investigated. A velocity potential is used to describe the fluid oscillations, and the Rayleigh–Ritz method has been extended to the case of fluid–structure interaction to obtain the solution of the coupled system.

1. INTRODUCTION

Vibrations of thin walled structures interacting with dense fluids are important in many engineering applications, for example in heat exchangers, nuclear plants, large storage tanks and rockets. Many studies are available on this topic from which circular cylindrical shells have received a large attention.

In the present study, free vibrations of circular cylindrical shells and tubes completely filled with a dense fluid and partially immersed in a different fluid (liquid) having a free surface are studied. Elastic shell constraints, varying from simply supported to clamped ends are assumed. This is an idealisation of a configuration that can be observed, for example, in off-shore structures or in steam condensers. Natural frequencies of tubes in similar situations are important for the computation of the critical velocity that gives the tube instability. The model developed is suitable to study flexural vibrations of both short and long shells (circular tubes). In fact, in some applications (e.g., steam condensers) the tubes are relatively long and can be conveniently studied as beams. It can also be noted that tubes in a steam condenser are usually expanded at the ends, so that they can be modelled with an elastic constraint

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with respect to the flexural slope, i.e., a constraint comprised between simply supported and clamped ends, depending on the assumed spring stiffness.

Different internal and external fluids are considered in this study. They are assumed to be stationary, inviscid and incompressible. The fluid outside the shell is assumed to be unlimited in the radial direction and limited in the vertical direction by a rigid bottom and a free surface. The effect of free surface waves is considered. Since the fluid inside the shell/tube is assumed to be stationary, the effect of fluid-flow inside the shell/tube is not studied. This effect has already been investigated for example by Païdoussis *et al.* [1], Païdoussis [2] and Weaver and Unny [3] in the case of fluid-filled shells and tubes, however, without considering free surface waves. Finally, in the present study, the velocity potential is used to describe the fluid oscillations, and the Rayleigh–Ritz method has been extended to the case of fluid–structure interaction [4] to obtain the solution of the coupled system.

Other studies can be related to the present one. Vibrations of beams in dense fluid have been studied, e.g., in references [5–7]. Vibrations of completely fluid-filled circular cylindrical shells have been investigated by Berry and Reissner [8]; their study was then extended to modes having more longitudinal half-waves by Lindholm *et al.* [9]. Shell vibrations in other configurations related to the present one have been studied, e.g., by Amabili [4], Amabili *et al.* [10], Au-Yang [11], Chiba [12], Gonçalves and Ramos [13], and Warburton [14]. However, it seems that no studies are available for circular cylindrical shells coupled to an external unbounded fluid, considering the effect of free surface waves and studying the coupling between sloshing modes (where the amplitude of the free surface waves is larger than the wall displacement) that are originated by fluid oscillations, and bulging (where the shell wall oscillates, thus moving the liquid) modes, that are due to the structure's elasticity. In order to lighten the text, in the following the word "shell" is sometimes also used to indicate tubes.

2. BASIC EQUATIONS

The Rayleigh–Ritz method [15] is applied to find the natural modes and frequencies of the simply supported shell, assuming the time variation to be harmonic. A cylindrical co-ordinate system $(O; x, \theta, r)$ is introduced with the origin O placed at the centre of the bottom end of the shell (Figure 1) and u, v, w are assumed to be the shell displacements in axial, circumferential and radial directions, respectively. The boundary conditions at the shell ends are

$$N_x = 0; \quad v = 0; \quad w = 0; \quad \text{at} \quad x = 0, L, \tag{1}$$

where N_x is the force per unit length in the x direction acting at the shell end. The flexural mode shapes w of the shell can be expressed as follows:

$$w(x, \theta) = \cos(n\theta) \sum_{s=1}^{\infty} q_s B_s \sin(s\pi x/L), \tag{2}$$

where L is the length of the shell, n is the number of circumferential waves, s is

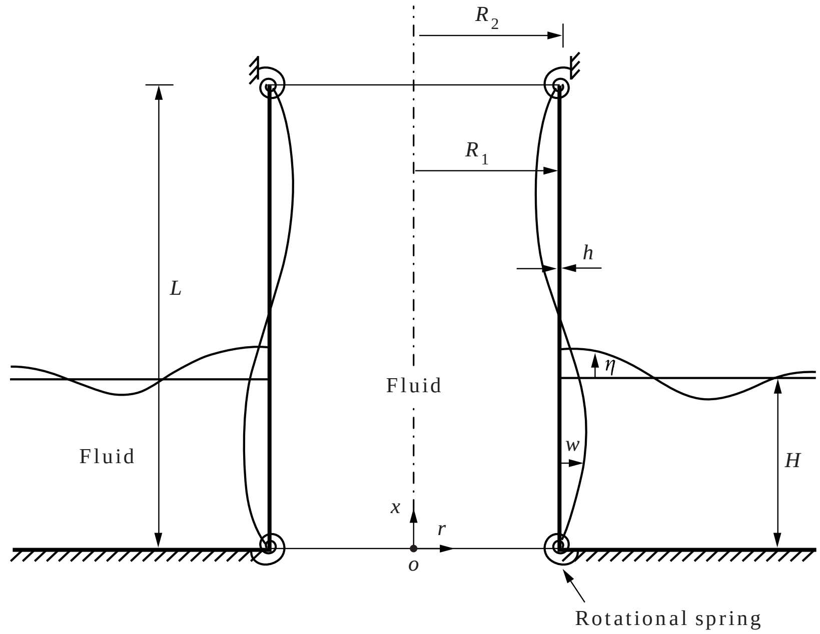

Figure 1. Geometry of the system and co-ordinates.

the number of axial half-waves, q_s are the parameters of the Ritz expansion and B_s is a constant dependent on the normalization criterion used; w is assumed to be positive outwards. The eigenvectors of the empty simply supported (also called shear diaphragm [16]) cylindrical shell are used as admissible functions. Obviously, for tube vibrations only modes having $n = 1$ are significant. In the present analysis, axisymmetric modes ($n = 0$) are not considered. The flexural displacement at time t is obtained multiplying w by the harmonic function $f(t) = \mathrm{e}^{\mathrm{i}\omega t}$, where $\mathrm{i} = \sqrt{-1}$ and ω is the radian frequency of vibration. Then, the following normalization is introduced:

$$\int_0^L B_s^2 \sin^2(s\pi x/L)\, \mathrm{d}x = 1, \tag{3}$$

and integration gives

$$B_s = B = \sqrt{2/L}. \tag{4}$$

In order to solve the problem, the kinetic and potential energies of the shell are evaluated. The reference kinetic energy T_T^* of the shell, neglecting the tangential and rotary inertia, is given by

$$T_T^* = \frac{1}{2}\rho_T h B^2 \chi_n \int_0^{2\pi}\int_0^L w^2\, \mathrm{d}x a\, \mathrm{d}\theta = \frac{1}{2}\rho_T a h \frac{L}{2} B^2 \chi_n \pi \sum_{s=1}^{\infty} q_s^2, \tag{5}$$

where h is the shell/tube thickness, a is the mean radius, ρ_T is the density of the shell material [kg/m^3] and $\chi_n = 1$ for $n \neq 1$ and $\chi_n = 2$ for beam bending ($n = 1$)

modes of long shells ($L/(sa) \geqslant 4$). In equation (5) the orthogonality of the sine function has been used. Neglecting tangential inertia is a very good approximation for thin shells. However, it is completely unacceptable for long shells in their beam bending ($n = 1$) modes [16]. For these modes the effective inertia is twice the quantity obtained with this approximation; this is the reason for introducing the parameter χ_n in equation (5).

It is now useful to note that the maximum potential energy of each mode of the empty shell is equal to the reference kinetic energy of the same mode multiplied by the squared circular frequency ω_s^2 of this mode. Moreover, due to the series expansion of the mode shape, the potential energy is the sum of the energies of each single component mode. As a consequence, the maximum potential energy of the shell may be expressed as

$$V_T = \frac{1}{2}\rho_T h a \frac{L}{2} B^2 \chi_n \pi \sum_{s=1}^{\infty} q_s^2 \omega_s^2, \tag{6}$$

where ω_s are the circular frequencies (rad/s) of the flexural modes of the simply supported shell that can be computed by using, for example, the Flügge theory [16] for shells or the classical formula for the free transverse vibrations of a simply supported beam [17] for tubes.

Elastic rotational springs of stiffness c (N m/m) are assumed to be distributed around each shell end (see Figure 1). The maximum potential energy V_S associated with these elastic springs is given by

$$\begin{aligned} V_S &= \frac{1}{2} c \int_0^{2\pi} \left[\left(\frac{\partial w}{\partial x}\right)^2_{x=0} + \left(\frac{\partial w}{\partial x}\right)^2_{x=L} \right] a \, \mathrm{d}\theta \\ &= \frac{caB^2\pi^2}{2L} \sum_{s=1}^{\infty} \sum_{j=1}^{\infty} q_s q_j s j [1 + (-1)^{s+j}]. \end{aligned} \tag{7}$$

It is to be noted that $c = 0$ gives simply supported ends and $c \to \infty$ gives clamped ends. In the computations, one takes a sufficiently high value of c to simulate a clamped end [4].

3. FLUID–STRUCTURE INTERACTION

The shell is considered completely filled with an inviscid and incompressible dense fluid and partially immersed in another dense fluid with a free surface orthogonal to the shell axis. The free surface is at a distance H from the rigid bottom (Figure 1). The ends of the shell are assumed to be open. Surface tension of the fluid and hydrostatic pressure effects are neglected in the present study.

For an incompressible and inviscid fluid, its deformation potential satisfies the Laplace equation

$$\nabla^2 \phi(x, \theta, r) = 0. \tag{8}$$

The deformation potential ϕ is related to the velocity potential $\tilde{\phi}$ by

$$\tilde{\phi}(x, \theta, r, t) = \mathrm{i}\omega\phi\, \mathrm{e}^{\mathrm{i}\omega t}, \tag{9}$$

which is assumed to be harmonic. The velocity of the fluid **v** is related to $\tilde{\phi}$ by $\mathbf{v} = -\mathrm{grad}\,\tilde{\phi}$. In the case studied, there are two different fluid domains. One is inside the shell/tube and one is outside.

3.1. FLUID INSIDE THE SHELL/TUBE

The Laplace equation (8) is solved with the boundary conditions

$$\phi_i = 0 \quad \text{at} \quad x = 0, L \qquad \text{and} \qquad (\partial\phi_i/\partial r)_{r=R_1} = -w, \tag{10a, b}$$

where ϕ_i is the deformation potential of the fluid inside the shell and R_1 is the inner radius of the shell. Moreover ϕ_i must be regular in the fluid domain. Equation (10a) states that the shell ends are open and equation (10b) ensures a contact between the shell wall and the fluid. Solution of Laplace equation satisfying equation (10a) and regularity is [8, 9]

$$\phi_i = \sum_{s=1}^{\infty} a_s \cos(n\theta)\mathrm{I}_n(s\pi r/L)\sin(s\pi x/L), \tag{11}$$

where I_n is the modified Bessel function of order n. Applying condition (10b), one obtains

$$a_s = -\frac{q_s B}{(s\pi/L)\mathrm{I}'_n(s\pi R_1/L)}, \tag{12}$$

where the prime indicates the derivative with respect to the argument. By using equations (11, 12) the fluid deformation potential ϕ_i at the shell–fluid internal interface is obtained as follows

$$(\phi_i)_{r=R_1} = -B\cos(n\theta)\sum_{s=1}^{\infty} q_s \sin(s\pi x/L)\frac{I_n(s\pi R_1/L)}{(s\pi/L)\mathrm{I}'_n(s\pi R_1/L)}. \tag{13}$$

The reference kinetic energy of the fluid inside the shell, by using Green's theorem [4, 18], is

$$\begin{aligned} T^*_{F_i} &= \frac{1}{2}\rho_{F_i}\int_0^L\int_0^{2\pi}\left(\phi_i\frac{\partial\phi_i}{\partial r}\right)_{r=R_1}\mathrm{d}xR_1\,\mathrm{d}\theta = -\frac{1}{2}\rho_{F_i}R_1\int_0^L\int_0^{2\pi}(\phi_i)_{r=R_1}w\,\mathrm{d}x\,\mathrm{d}\theta \\ &= \frac{1}{2}\rho_{F_i}R_1\frac{L}{2}B^2\pi\sum_{s=1}^{\infty}q_s^2\frac{I_n(s\pi R_1/L)}{(s\pi/L)\mathrm{I}'_n(s\pi R_1/L)}, \end{aligned} \tag{14}$$

where ρ_{F_i} is the mass density of the fluid inside the shell. A discussion of the effect of the shell radius on equation (14) can be found in Appendix A. No potential energy is associated with the fluid inside the shell as a consequence that it is incompressible and that it does not present a free surface.

3.2. FLUID OUTSIDE THE SHELL/TUBE

The fluid domain outside the shell is not limited in the radial direction; in the x direction it is limited by a rigid surface at $x = 0$ and the free surface at $x = H$. The fluid deformation potential ϕ_o, using the principle of superposition, can be divided into

$$\phi_o = \phi_B + \phi_S, \tag{15}$$

where ϕ_B describes the potential of the fluid associated with bulging modes of the shell considering a zero dynamic pressure on the undisturbed free surface and ϕ_S is due to the sloshing (oscillations) of the fluid considering the shell as rigid.

The boundary conditions imposed to the liquid for the two complementary boundary value problems are

$$\left(\frac{\partial \phi_B}{\partial x}\right)_{x=0} = 0, \quad \left(\frac{\partial \phi_B}{\partial r}\right)_{r=R_2} = -w, \quad (\phi_B)_{x=H} = 0, \quad \lim_{r\to\infty} \phi_B = \lim_{r\to\infty} (\partial \phi_B/\partial r) = 0, \tag{16a–d}$$

and

$$\left(\frac{\partial \phi_S}{\partial x}\right)_{x=0} = 0, \quad \left(\frac{\partial \phi_S}{\partial r}\right)_{r=R_2} = 0, \quad g\left(\frac{\partial \phi_o}{\partial x}\right)_{x=H} = \omega^2 (\phi_o)_{x=H}, \quad \lim_{r\to\infty} \phi_S = \lim_{r\to\infty} (\partial \phi_S/\partial r) = 0. \tag{17a–d}$$

In equations (16) and (17), R_2 is the outer radius of the shell. By using equations (15) and (16c), the linearized free surface condition [18], equation (17c), can be rewritten as

$$g\left[\frac{\partial(\phi_B + \phi_S)}{\partial x}\right]_{x=H} = \omega^2 (\phi_S)_{x=H}, \tag{18}$$

where g is the gravity acceleration.

The Rayleigh quotient [4, 19] for the coupled fluid-structure system studied, is given by:

$$\omega^2 = (V_T + V_S + V_{F_o})/(T_T^* + T_{F_i}^* + \tilde{T}_{F_o}^*). \tag{19}$$

The only terms that remain to be computed in equation (19) are the reference kinetic energy of the fluid outside the shell, $\tilde{T}_{F_o}^*$, and its maximum potential energy, V_{F_o}, related to the free surface waves of the fluid itself. By using Green's theorem for harmonic functions [4, 18], the reference kinetic energy of the fluid outside the shell can be transformed into

$$\tilde{T}_{F_o}^* = \frac{1}{2}\rho_{F_o} \iint_{S_T+S_F} \phi_o \frac{\partial \phi_o}{\partial z}\, \mathrm{d}S, \tag{20}$$

where ρ_{F_o} is the mass density of external fluid, z is the direction normal to any point on the boundary surface S of the fluid domain and is pointed outwards, $S = S_T + S_F$, where S_T is the shell lateral external surface and S_F is the free liquid surface (no contribution to $\tilde{T}^*_{F_o}$ is given by integration over the rigid bottom). The simplified reference kinetic energy $T^*_{F_o}$ of the fluid outside the shell is also defined as

$$T^*_{F_o} = \frac{1}{2}\rho_{F_o} \iint_{S_T} \phi_o \frac{\partial \phi_o}{\partial z}\, \mathrm{d}S = \frac{1}{2}\rho_{F_o} \iint_{S_T} (\phi_B + \phi_S) w \, \mathrm{d}S = T^*_{F_B} + T^*_{F_S}. \tag{21}$$

The maximum potential energy V_F of the free surface waves of the fluid is given by [4]

$$V_{F_o} = \frac{1}{2}\rho_{F_o} g \iint_{S_F} \frac{\partial \phi_o}{\partial z}\frac{\partial \phi_o}{\partial z}\, \mathrm{d}S = \frac{1}{2}\rho_{F_o}\omega^2 \iint_{S_F} \phi_o \frac{\partial \phi_o}{\partial z}\, \mathrm{d}S, \tag{22}$$

where the second equality is obtained by using the free surface condition, equation (17c). It is interesting to observe that, by using equations (20) and (22), the Rayleigh quotient can be rewritten in the following simplified form:

$$\omega^2 = (V_T + V_S)/(T^*_T + T^*_{F_i} + T^*_{F_o}), \tag{23}$$

where the potential energy V_{F_o} does not appear. Furthermore, it is no longer necessary to integrate the quantity $\phi_o(\partial\phi_o/\partial z)$ over the free surface of the fluid S_F. In conclusion, only the additional term $T^*_{F_o}$ due to the external fluid must still be computed and is given by two terms, as shown in equation (21).

3.2.1. *Fluid deformation potential related to bulging modes*

In this section, the deformation potential of the fluid related to bulging modes of the shell is investigated. The fluid deformation potential ϕ_B is assumed to be of the form

$$\phi_B = \sum_{s=1}^{\infty} q_s \Phi_s. \tag{24}$$

The functions Φ_s are given by

$$\Phi_s(x, \theta, r) = \sum_{m=1}^{\infty} A_{sm} \mathrm{K}_n\left(\frac{2m-1}{2}\pi\frac{r}{H}\right) \cos\left(\frac{2m-1}{2}\pi\frac{x}{H}\right) \cos(n\theta), \tag{25}$$

where A_{sm} are coefficients depending on the integers s and m, H is the fluid level and K_n is the modified Bessel function of order n. Functions Φ_s satisfy the Laplace equation and the two boundary conditions given in equations (16b, c); moreover they satisfy the Sommerfeld radiation condition (16d). The condition given in equation (16a) is used to compute the coefficients A_{sm}:

$$\sum_{m=1}^{\infty} A_{sm} \frac{(2m-1)\pi}{2H} \mathrm{K}'_n\left(\frac{2m-1}{2}\pi\frac{R_2}{H}\right) \cos\left(\frac{2m-1}{2}\pi\frac{x}{H}\right) = -B \sin\left(s\pi\frac{x}{L}\right). \tag{26}$$

Equation (26) must be satisfied for all values of $0 \leqslant x \leqslant H$. If this equation is multiplied by $\cos(\frac{1}{2}(2j-1)(\pi x/H))$ and then integrated between 0 and H, using the well-known properties of the orthogonal trigonometric functions, the following equation is obtained

$$\Phi_s = \sum_{m=1}^{\infty} \frac{-4B}{(2m-1)\pi} \sigma_{sm} \frac{\mathrm{K}_n\left(\dfrac{2m-1}{2}\pi \dfrac{r}{H}\right)}{\mathrm{K}'_n\left(\dfrac{2m-1}{2}\pi \dfrac{R_2}{H}\right)} \cos\left(\frac{2m-1}{2}\pi \frac{x}{H}\right) \cos(n\theta), \tag{27}$$

where

$$\sigma_{sm} = \frac{\dfrac{s}{L} + (-1)^m \dfrac{2m-1}{2H} \sin\left(s\pi \dfrac{H}{L}\right)}{\left(\dfrac{s^2}{L^2} - \dfrac{4m^2-4m+1}{4H^2}\right)\pi} \quad \text{if} \quad s \neq \frac{2m-1}{2}\frac{L}{H}, \tag{28a}$$

or

$$\sigma_{sm} = \frac{L}{2s\pi} \quad \text{if} \quad s = \frac{2m-1}{2}\frac{L}{H}. \tag{28b}$$

Therefore, the term $T^*_{F_B}$ of the reference kinetic energy of the fluid is given by

$$\begin{aligned} T^*_{F_B} &= \frac{1}{2}\rho_{Fo} \int_0^{2\pi} \int_0^H (\phi_B)_{r=R_2} w R_2 \, \mathrm{d}\theta \, \mathrm{d}x \\ &= -\frac{1}{2}\rho_{F_o} B^2 R_2 \pi \sum_{s=1}^{\infty}\sum_{j=1}^{\infty} q_s q_j \sum_{m=1}^{\infty} \frac{4\sigma_{sm}\sigma_{jm}}{(2m-1)\pi} \frac{\mathrm{K}_n\left(\dfrac{2m-1}{2}\pi \dfrac{R_2}{H}\right)}{\mathrm{K}'_n\left(\dfrac{2m-1}{2}\pi \dfrac{R_2}{H}\right)}. \end{aligned} \tag{29}$$

3.2.2. *Fluid deformation potential related to sloshing modes*

The fluid deformation potential ϕ_S due to the sloshing can be written in the form

$$\begin{aligned} \phi_S = \sum_{m=1}^{\infty} \Bigg[& F_m \mathrm{J}_n\left(\varepsilon_m \frac{r}{R_2}\right) \cosh\left(\varepsilon_m \frac{x}{R_2}\right) \Big/ \cosh\left(\varepsilon_m \frac{H}{R_2}\right) \\ & + G_m \mathrm{Y}_n\left(\tilde{\varepsilon}_m \frac{r}{R_2}\right) \cosh\left(\tilde{\varepsilon}_m \frac{x}{R_2}\right) \Big/ \cosh\left(\tilde{\varepsilon}_m \frac{H}{R_2}\right)\Bigg] \cos(n\theta), \end{aligned} \tag{30}$$

where F_m and G_m are the parameters of the Ritz expansion of the sloshing modes, J_n and Y_n are the Bessel functions of order n and ε_m and $\tilde{\varepsilon}_m$ are solutions of the following equations

$$\mathrm{J}'_n(\varepsilon_m) = 0, \quad \mathrm{Y}'_n(\tilde{\varepsilon}_m) = 0, \qquad \text{for} \qquad m = 1, \ldots \infty. \tag{31a, b}$$

Constants $\cosh(\varepsilon_m H/R_2)$ and $\cosh(\tilde{\varepsilon}_m H/R_2)$ are not necessary in equation (30); however, they are useful to obtain a well-conditioned mass matrix of the system in the Galerkin equation that is obtained by applying the Rayleigh–Ritz method.

The term $T^*_{F_s}$ of the kinetic energy of the fluid due to sloshing is

$$\begin{aligned} T^*_{F_S} &= \frac{1}{2}\rho_{F_o}\int_0^{2\pi}\int_0^H (\phi_S)_{r=R_2} w R_2 \,\mathrm{d}x\,\mathrm{d}\theta \\ &= \frac{1}{2}\rho_{F_o} R_2^2 \pi B \sum_{s=1}^{\infty}\sum_{m=1}^{\infty} q_s [F_m \mathrm{J}_n(\varepsilon_m)\gamma_{sm}/\cosh(\varepsilon_m H/R_2) \\ &\qquad + G_m \mathrm{Y}_n(\tilde{\varepsilon}_m)\tilde{\gamma}_{sm}/\cosh(\tilde{\varepsilon}_m H/R_2)], \end{aligned} \tag{32}$$

where

$$\begin{aligned} \gamma_{sm} &= \frac{1}{R_2}\int_0^H \cosh\left(\varepsilon_m \frac{x}{R_2}\right)\sin\left(\frac{s\pi x}{L}\right)\mathrm{d}x \\ &= \frac{\dfrac{s\pi R_2}{L} - \dfrac{s\pi R_2}{L}\cos\left(\dfrac{s\pi H}{L}\right)\cosh\left(\varepsilon_m \dfrac{H}{R_2}\right) + \varepsilon_m \sin\left(\dfrac{s\pi H}{L}\right)\sinh\left(\varepsilon_m \dfrac{H}{R_2}\right)}{\varepsilon_m^2 + \dfrac{s^2\pi^2 a^2}{L^2}}, \end{aligned} \tag{33}$$

and $\tilde{\gamma}_m$ is obtained from γ_m by substituting ε_m with $\tilde{\varepsilon}_m$ in equation (33). The potential ϕ_S satisfies equations (17a, b, d). Now the free surface condition, equation (18), must be applied. By using equations (27, 30) and eliminating $\cos(n\theta)$, it gives

$$\begin{aligned} &-\frac{2B}{H}\sum_{s=1}^{\infty} q_s \sum_{k=1}^{\infty}(-1)^k \sigma_{sk}\mathrm{K}_n\left(\frac{2k-1}{2}\pi\frac{r}{H}\right)\Big/\mathrm{K}_n'\left(\frac{2k-1}{2}\pi\frac{R_2}{H}\right) \\ &+\sum_{m=1}^{\infty}\left[\frac{\varepsilon_m}{R_2}F_m \mathrm{J}_n\left(\varepsilon_m\frac{r}{R_2}\right)\tanh\left(\varepsilon_m\frac{H}{R_2}\right) + \frac{\tilde{\varepsilon}_m}{R_2}G_m \mathrm{Y}_n\left(\tilde{\varepsilon}_m\frac{r}{R_2}\right)\tanh\left(\tilde{\varepsilon}_m\frac{H}{R_2}\right)\right] \\ &= \frac{\omega^2}{g}\sum_{m=1}^{\infty}\left[F_m \mathrm{J}_n\left(\varepsilon_m\frac{r}{R_2}\right) + G_m \mathrm{Y}_n\left(\tilde{\varepsilon}_m\frac{r}{R_2}\right)\right]. \end{aligned} \tag{34}$$

Equation (34) must be satisfied for all values of $R_2 \leqslant r < \infty$. In the case of a rigid shell, natural frequencies of sloshing modes are immediately found. They are $\omega_m^2 = g(\varepsilon_m/R_2)\tanh(\varepsilon_m H/R_2)$ and $\omega_m^2 = g(\tilde{\varepsilon}_m/R_2)\tanh(\tilde{\varepsilon}_m H/R_2)$. In contrast, the variable $\rho = R_2/r$, $0 < \rho \leqslant 1$, is introduced and the following Fourier–Bessel expansions [20, 21] to solve the problem for a flexible shell are used

$$\mathrm{J}_n(\varepsilon_m/\rho) = \sum_{s=1}^{\infty} a_{sm}\mathrm{J}_n(\varepsilon_s\rho), \tag{35}$$

$$\mathrm{Y}_n(\tilde{\varepsilon}_m/\rho) = \sum_{s=1}^{\infty} b_{sm}\mathrm{J}_n(\varepsilon_s\rho), \tag{36}$$

$$\mathrm{K}_n\left(\frac{2k-1}{2}\frac{\pi}{H}\frac{R_2}{\rho}\right)=\sum_{s=1}^{\infty}c_{sk}\mathrm{J}_n(\varepsilon_s\rho), \tag{37}$$

where

$$a_{sm}=\frac{2\varepsilon_s^2}{(\varepsilon_s^2-n^2)\mathrm{J}_n^2(\varepsilon_s)}\int_0^1 x\,\mathrm{J}_n\left(\frac{\varepsilon_m}{x}\right)\mathrm{J}_n(\varepsilon_s x)\,\mathrm{d}x, \tag{38}$$

$$b_{sm}=\frac{2\varepsilon_s^2}{(\varepsilon_s^2-n^2)\mathrm{J}_n^2(\varepsilon_s)}\int_0^1 x\,\mathrm{Y}_n\left(\frac{\tilde{\varepsilon}_m}{x}\right)\mathrm{J}_n(\varepsilon_s x)\,\mathrm{d}x, \tag{39}$$

$$c_{sk}=\frac{2\varepsilon_s^2}{(\varepsilon_s^2-n^2)\mathrm{J}_n^2(\varepsilon_s)}\int_0^1 x\,\mathrm{K}_n\left(\frac{2k-1}{2}\frac{\pi}{H}\frac{R_2}{x}\right)\mathrm{J}_n(\varepsilon_s x)\,\mathrm{d}x. \tag{40}$$

The functions given in equations (35, 36) are highly oscillating for ρ close to zero. However, it was verified that the series expansion converges to the function in the entire region little away from zero. Inserting equations (35–37) into equation (34), the following set of algebraic equations is obtained:

$$\begin{aligned}&-\frac{2B}{H}\sum_{s=1}^{\infty}q_s\sum_{k=1}^{\infty}(-1)^k\frac{\sigma_{sk}}{\mathrm{K}'\left(\frac{2k-1}{2}\pi\frac{R_2}{H}\right)}c_{ik}\\&+\sum_{m=1}^{\infty}\left[\frac{\varepsilon_m}{R_2}F_m a_{im}\tanh\left(\varepsilon_m\frac{H}{R_2}\right)+\frac{\tilde{\varepsilon}_m}{R_2}G_m b_{im}\tanh\left(\tilde{\varepsilon}_m\frac{H}{R_2}\right)\right]\\&=\frac{\omega^2}{g}\sum_{m=1}^{\infty}[F_m a_{im}+G_m b_{im}],\quad\text{for}\quad i=1,\ldots\infty\,.\end{aligned} \tag{41}$$

Equation (34) can also be used to obtain a set of algebraic equations in a different way. In fact, it is possible to compute all functions of r at different values r_i, $R_2\leqslant r_i<\infty$, in order to formally have the same equation (41), where the following expressions substitute equations (38), (39) and (40):

$$a_{im}=\mathrm{J}_n\left(\varepsilon_m\frac{r_i}{R_2}\right),\quad b_{im}=\mathrm{Y}_n\left(\tilde{\varepsilon}_m\frac{r_i}{R_2}\right),\quad c_{ik}=\mathrm{K}_n\left(\frac{2k-1}{2}\pi\frac{r_i}{H}\right). \tag{42a–c}$$

Finally, the height η of free surface waves can be computed by using the following expression [18]

$$\eta(t)=\frac{1}{g}\left(\frac{\partial\tilde{\phi}}{\partial t}\right)_{x=H}=-\frac{\omega^2}{g}(\phi_S)_{x=H}. \tag{43}$$

4. FREQUENCY EQUATION

For the numerical calculation of the natural frequencies and the parameters of the Ritz expansion of modes, only N terms in the expansion of w, equation (1), and $2\bar{N}$ in the expansion of ϕ_S, equation (30), are considered, where N and $\bar{N}$ must be chosen large enough to give the required accuracy to the solution. Thus, all the energies are given by finite summations. Here it is convenient to introduce a vectorial notation. The vector $\mathbf{q}$ of the parameters of the Ritz expansions is defined by

$$\mathbf{q}^{\mathrm{T}} = \{\{\mathbf{q}\}^{\mathrm{T}}, \{\mathbf{F}\}^{\mathrm{T}}, \{\mathbf{G}\}^{\mathrm{T}}\}, \quad \{\mathbf{q}\}^{\mathrm{T}} = \{q_1, \dots\ q_N\},$$
$$\{\mathbf{F}\}^{\mathrm{T}} = \{F_1, \dots\ F_{\bar{N}}\}, \quad \{\mathbf{G}\}^{\mathrm{T}} = \{G_1, \dots\ G_{\bar{N}}\}. \tag{44}$$

The maximum potential energy of the shell/tube, equation (6), becomes

$$V_T = \frac{1}{2}\pi\{\mathbf{q}\}^{\mathrm{T}}\mathbf{K}_T\{\mathbf{q}\}. \tag{45}$$

The elements of the diagonal matrix $\mathbf{K}_T$ are given by

$$[\mathbf{K}_T]_{sj} = \delta_{sj}\rho_T ha\frac{L}{2}B^2\chi_n\omega_s^2, \quad s, j = 1, \dots, N, \tag{46}$$

and δ_{sj} is the Kronecker delta.

The maximum potential energy stored in the elastic spring constraints, using equation (7), is given by

$$V_S = \frac{1}{2}\pi\{\mathbf{q}\}^{\mathrm{T}}\mathbf{K}_S\{\mathbf{q}\}, \tag{47}$$

where the elements of the matrix $\mathbf{K}_S$ are

$$[\mathbf{K}_S]_{sj} = c(a/L)\pi B^2 sj[1 + (-1)^{s+j}], \quad s, j = 1, \dots, N. \tag{48}$$

The reference kinetic energy of the shell/tube, equation (5), may be rewritten as

$$T_T^* = \frac{1}{2}\pi\{\mathbf{q}\}^{\mathrm{T}}\mathbf{M}_T\{\mathbf{q}\}, \tag{49}$$

where

$$\mathbf{M}_T = \rho_T ha(L/2)B^2\chi_n\mathbf{I}, \tag{50}$$

and $\mathbf{I}$ is the $N \times N$ identity matrix.

The simplified reference kinetic energy of the fluid, that was previously divided into one contribution due to the fluid inside the shell and two different contributions due to the fluid outside, equation (21), can be rewritten as

$$T_L^* = \frac{1}{2}\pi\mathbf{q}^{\mathrm{T}}\mathbf{M}_F\mathbf{q}, \tag{51}$$

where $\mathbf{M}_F$ is a symmetric partitioned matrix of dimension $(N + 2\bar{N}) \times (N + 2\bar{N})$:

$$\mathbf{M}_F = \left[\begin{array}{c|c|c} \mathbf{M}_B & \mathbf{M}_{S1} & \mathbf{M}_{S2} \\ \hline [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] \end{array}\right], \tag{52}$$

in which the submatrix $\mathbf{M}_B$ of dimension $N \times N$ is given by the contribution of $T^*_{F_i}$ and $T^*_{F_B}$

$$\mathbf{M}_B = \mathbf{M}_i + \mathbf{M}_o. \tag{53}$$

The elements of the submatrix $\mathbf{M}_B$ due to $T^*_{F_i}$ according to equation (14), are given by

$$[\mathbf{M}_i]_{sj} = \delta_{sj}\rho_{F_i} R_1 \frac{L}{2} B^2 \frac{\mathrm{I}_n(s\pi R_1/L)}{(s\pi/L)\mathrm{I}'_n(s\pi R_1/L)}, \quad \text{for} \quad s, j = 1, \ldots, N, \tag{54}$$

where δ_{sj} is the Kronecker delta. The elements of the submatrix $\mathbf{M}_o$ due to $T^*_{F_B}$, according to equation (29), are given by

$$[\mathbf{M}_o]_{sj} = -\rho_{F_o} R_2 B^2 \sum_{m=1}^{\infty} \frac{4\sigma_{sm}\sigma_{jm}}{(2m-1)\pi} \frac{\mathrm{K}_n\left(\dfrac{2m-1}{2}\pi\dfrac{R_2}{H}\right)}{\mathrm{K}'_n\left(\dfrac{2m-1}{2}\pi\dfrac{R_2}{H}\right)}, \quad \text{for} \quad s, j = 1, \ldots, N, \tag{55}$$

where σ_{sm} are defined in equations (28a, b).

The elements of the submatrices $\mathbf{M}_{S1}$ and $\mathbf{M}_{S2}$ of dimension $N \times \bar{N}$, according to equation (32), are

$$[\mathbf{M}_{S1}]_{sm} = \rho_{F_o} R_2^2 B\, \mathrm{J}_n(\varepsilon_m)\gamma_{sm}/\cosh(\varepsilon_m H/R_2), \quad \text{for} \quad s = 1, \ldots, N \quad \text{and} \quad m = 1, \ldots, \bar{N}, \tag{56}$$

$$[\mathbf{M}_{S2}]_{sm} = \rho_{F_o} R_2^2 B\, \mathrm{Y}_n(\tilde{\varepsilon}_m)\tilde{\gamma}_{sm}/\cosh(\tilde{\varepsilon}_m H/R_2), \quad \text{for} \quad s = 1, \ldots, N \quad \text{and} \quad m = 1, \ldots, \bar{N}. \tag{57}$$

The free surface condition, equation (41), can be rewritten in the following form:

$$\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}\left\{\begin{array}{c} \{\mathbf{q}\} \\ \{\mathbf{F}\} \\ \{\mathbf{G}\} \end{array}\right\} = \omega^2\{[\mathbf{0}], \mathbf{H}_1, \mathbf{H}_2\}\left\{\begin{array}{c} \{\mathbf{q}\} \\ \{\mathbf{F}\} \\ \{\mathbf{G}\} \end{array}\right\}, \tag{58}$$

where $\mathbf{E}_1$ has dimension $2\bar{N} \times N$ and $\mathbf{E}_2$, $\mathbf{E}_3$, $\mathbf{H}_1$ and $\mathbf{H}_2$ have dimension $2\bar{N} \times \bar{N}$; these matrices are given by

$$[\mathbf{E}_1]_{ij} = -g\frac{2B}{H}\sum_{k=1}^{\infty} \sigma_{jk} c_{ik} (-1)^k / \mathrm{K}'_n\left(\frac{2k-1}{2}\pi\frac{R_2}{H}\right), \quad \text{for} \quad i = 1, \ldots, 2\bar{N} \quad \text{and}$$

$$j = 1, \dots N, \tag{59}$$

$$[\mathbf{E}_2]_{ij} = g a_{ij}(\varepsilon_j/R_2)\tanh(\varepsilon_j H/R_2), \quad \text{for } i = 1, \dots 2\bar{N} \quad \text{and} \quad j = 1, \dots \bar{N}, \tag{60}$$

$$[\mathbf{E}_3]_{ij} = g b_{ij}(\tilde{\varepsilon}_j/R_2)\tanh(\tilde{\varepsilon}_j H/R_2), \quad \text{for } i = 1, \dots 2\bar{N} \quad \text{and} \quad j = 1, \dots \bar{N}, \tag{61}$$

$$[\mathbf{H}_1]_{ij} = a_{ij}, \quad \text{for} \quad i = 1, \dots 2\bar{N} \quad \text{and} \quad j = 1, \dots \bar{N}, \tag{62}$$

$$[\mathbf{H}_2]_{ij} = b_{ij}, \quad \text{for} \quad i = 1, \dots 2\bar{N} \quad \text{and} \quad j = 1, \dots \bar{N}. \tag{63}$$

The values of the vector $\mathbf{q}$ of the parameters of the Ritz expansion are determined in order to render the Rayleigh quotient of equation (23) stationary [15], by also inserting in the eigenvalue problem the free surface condition that determines the value of the coefficients F_m and G_m [4, 10, 12, 13]. Then the following Galerkin equation is obtained:

$$\begin{bmatrix} \mathbf{K}_T + \mathbf{K}_S & [\mathbf{0}] & [\mathbf{0}] \\ \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \end{bmatrix} \mathbf{q} - \Lambda^2 \begin{bmatrix} \mathbf{M}_T + \mathbf{M}_B & \mathbf{M}_{S1} & \mathbf{M}_{S2} \\ [\mathbf{0}] & \mathbf{H}_1 & \mathbf{H}_2 \end{bmatrix} \mathbf{q} = 0, \tag{64}$$

where Λ is the circular frequency of the shell/tube filled and partially submerged with fluid. Equation (64) gives a linear eigenvalue problem for a real, non-symmetric matrix.

The pressure exerted by the fluid at a point of the shell/tube wall can be computed by using the linearized Bernoulli equation:

$$(p)_{point} = \rho_F(\partial\tilde{\phi}/\partial t)_{point} = -\rho_F\omega^2(\phi)_{point}\,\mathrm{e}^{\mathrm{i}\omega t}, \tag{65}$$

where ρ_F and ϕ are relative to the fluid inside or outside the shell.

5. NUMERICAL RESULTS

Numerical solutions have been obtained by using the software *Mathematica* [22] to compute matrices and solve the eigenvalue problem associated with equation (64). In particular, 10 terms have been used in the expansion of shell modes and four terms in the expansion of the sloshing potential. These are enough to give a good accuracy for studied cases. In fact, the eigenvalues quickly converge (from above) to the actual ones increasing the number of terms used in the expansion. Table 1 shows the convergence of the solution with the number N of terms in the expansion of w (first case studied).

The first case studied is a water-filled, simply supported shell partially in contact with external water to the level $H = 1$ m. The following dimensions and material properties are taken: $a = 0{\cdot}25$ m, $L = 2$ m, $h = 1$ mm, $\rho_{F_i} = \rho_{F_o} = 1000$ kg/m^3, $\rho_T = 7850$ kg/m^3, $E = 206$ GPa and $\nu = 0{\cdot}3$. They correspond to a very thin steel shell in water. The natural frequencies of the first

TABLE 1

Natural frequencies (Hz) of the first four bulging modes of the steel shell, obtained with different number N of terms in the Rayleigh–Ritz expansions of w; n = 2 and $\bar{N}$ = 2

N	1st mode	2nd mode	3rd mode	4th mode
2	22·39	82·72	–	–
4	22·39	82·33	162·42	259·38
6	22·39	82·33	162·24	254·63
8	22·39	82·33	162·21	254·62
10	22·39	82·33	162·20	254·59

three modes of the shell *in vacuo* are given in Figure 2 versus the number n of circumferential waves. *In vacuo*, the 1st mode corresponds to $m = 1$, the 2nd mode to $m = 2$ and the 3rd mode to $m = 3$, where m is the number of axial half-waves. Figure 2 shows that the fundamental mode of the studied shell has $(n, m) = (3, 1)$.

The natural frequencies of sloshing and bulging modes of the water-filled shell in contact with external water up to $H = 1$ m are shown in Figures 3 and 4, respectively. The natural frequencies of sloshing modes increase with n, while the fundamental bulging mode of the system has $n = 4$. It is important to note that, in this case, the number m of axial half-waves has no more importance, as a consequence that natural modes are given by a superposition of sine functions having different m values. The effect of fluid is to decrease largely the natural frequencies of bulging modes, which are modes originated by the elasticity of the structure, and to introduce in the system modes due to oscillations (sloshing) of external liquid having very low natural frequencies.

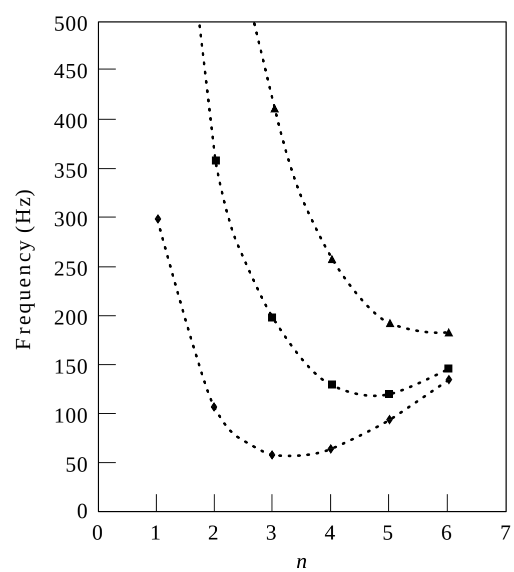

Figure 2. Natural frequencies of the empty shell studied as a function of the number of nodal diameters n. ◆, 1st mode; ■, 2nd mode; ▲, 3rd mode.

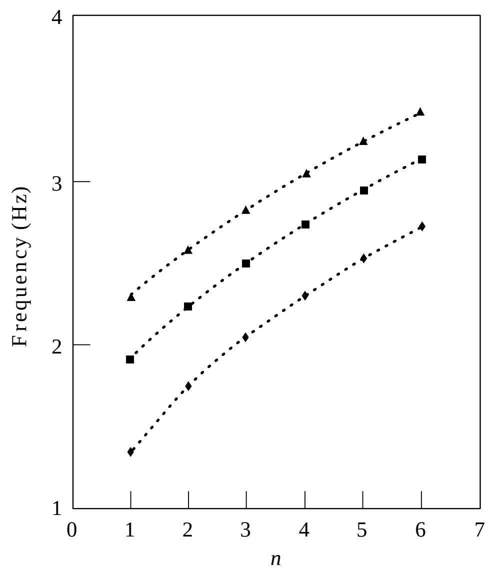

Figure 3. Natural frequencies of sloshing modes of the water-filled shell partially immersed up to $H = 1$ m as a function of the number of nodal diameters, n. ◆, 1st mode; ■, 2nd mode; ▲, 3rd mode.

The second case studied is an empty, simply supported shell partially in contact with external water. The following dimensions and material properties are taken: $a = 1{\cdot}27$ m, $L = 1$ m, $h = 3$ mm, $\rho_{F_o} = 1000$ kg/m^3, $\rho_T = 3656$ kg/m^3, $E = 68{\cdot}65$ GPa and $\nu = 0{\cdot}3$. They correspond to a squat aluminium shell in water. The natural frequencies of the first four modes of the shell *in vacuo*,

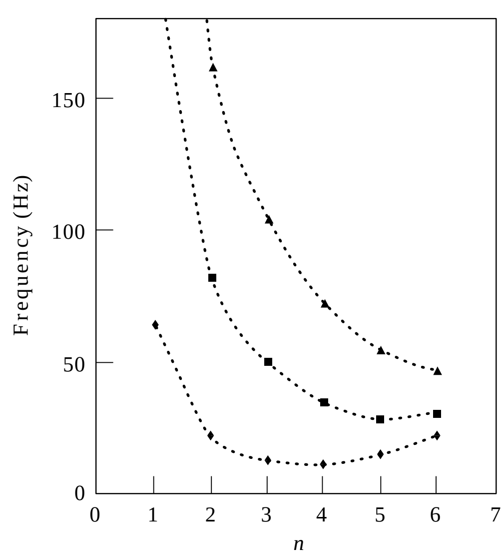

Figure 4. Natural frequencies of bulging modes of the water-filled shell partially immersed up to $H = 1$ m as a function of the number of nodal diameters, n. ◆, 1st mode; ■, 2nd mode; ▲, 3rd mode.

according with Flügge theory of shells, for $n = 4$ are: 265·1, 432·8, 491·6 and 521·3 Hz. The effect of level H of external water for bulging modes having $n = 4$ is investigated in Figure 5. Bulging mode shapes for $H = 0{\cdot}5$ ($n = 4$) are shown in Figure 6, where it is interesting to note that significant waves (represented in the same scale of shell displacement) on the water surface are associated to the first bulging modes of the system. This figure shows a section of the shell along its longitudinal axis. Mode shapes are symmetrical with respect to the shell axis since n is even. The complex shape of the radial shell displacement w is also clearly visible in the figure.

The free surface waves are in-phase with the shell oscillation. These waves, at the shell–water interface, present a local maximum or minimum depending on the shell mode shape. In particular, the shell is shown with inwards displacement in Figure 6(a), so that the fluid has moved in the inward direction and there is a minimum at the shell–water interface. Similar phenomena are observed for the other modes shown in Figures 6(b–d).

It is interesting to note that in the studied cases there is a significant separation between bulging and sloshing frequencies, so that the coupling between the two families of modes is quite weak. However, for extremely thin and flexible shells a much more significant coupling is expected.

6. CONCLUSIONS

Bulging modes of thin shells are largely affected by the presence of internal and external dense fluids. Both natural frequencies and mode shapes are modified by the fluid–structure interaction. Moreover, external fluid presents a free surface, so that it introduces in the system a second family of modes, the sloshing modes, characterized by low natural frequencies.

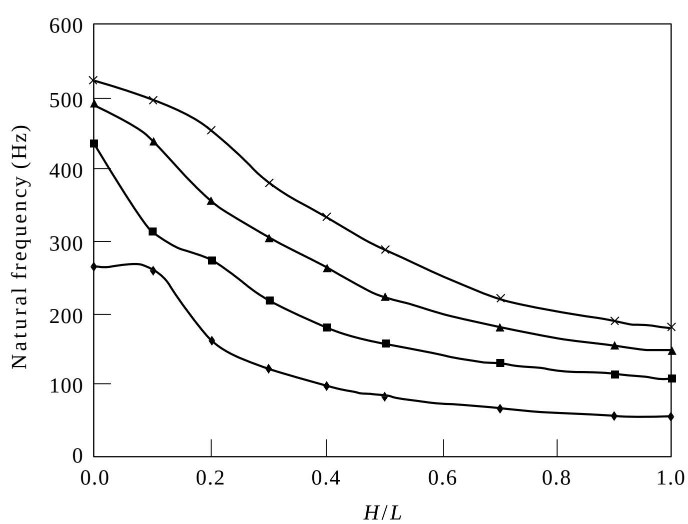

Figure 5. Natural frequency of bulging modes of the empty shell partially immersed in water as a function of the level H/L for $n = 4$. ◆, 1st mode; ■, 2nd mode; ▲, 3rd mode; × 4th mode.

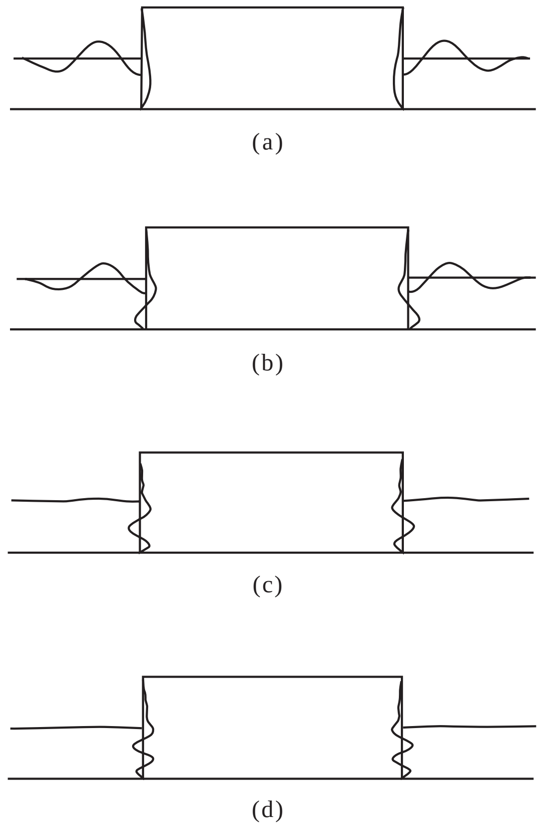

Figure 6. Mode shapes of the first four bulging modes of the empty shell partially immersed in water up to $H = 0{\cdot}5$ m for $n = 4$. (a) 1st mode, frequency 82·1 Hz; (b) 2nd mode, frequency 155·3 Hz; (c) 3rd mode, frequency 221·6 Hz; (d) 4th mode, frequency 286·4 Hz.

It seems that no studies are available for circular cylindrical shells coupled to an external unbounded fluid, considering the effect of free surface waves. An unbounded fluid domain, such as the one studied, can be simulated with difficulty by standard finite element programs. In contrast, the Rayleigh–Ritz method, employed in the present study, allows a fast and reliable solution to the problem to be obtained.

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APPENDIX A: EFFECT OF THE RADIUS ON THE INERTIA OF INTERNAL FLUID

It is interesting to note that, for small y values, $\mathrm{I}_n(y)/\mathrm{I}'_n(y) \cong y$, whereas for large y values, $\mathrm{I}_n(y)/\mathrm{I}'_n(y) \cong 1$. Therefore, the reference kinetic energy of the fluid inside the shell/tube, equation (14), for small R_1/L is given by

$$T^*_{F_i} = \frac{1}{2}\rho_{F_i} R_1^2 \pi \frac{L}{2} B^2 \sum_{s=1}^{\infty} q_s^2. \tag{A1}$$

Expression (A1) is easily related to the inertia of the whole fluid mass subjected to a displacement being only a function of the axial co-ordinate x. Therefore, it is verified that, for small R_1/L values, the system composed of shell/tube and fluid has a virtual mass given by the mass of the shell plus the mass of the contained fluid.

APPENDIX B: NOMENCLATURE

a	mean shell radius
B	normalization coefficient
c	stiffness of rotational springs
E	Young's modulus
g	gravity acceleration
h	shell thickness
H	fluid level
I_n	modified Bessel function of order n
J_n	Bessel function of order n
K_n	modified Bessel function of order n
L	shell length
n	number of circumferential waves
r	radial co-ordinate
R_1	internal shell radius
R_2	external shell radius
s	number of axial half-waves
u	axial shell displacement
v	circumferential shell displacement
w	radial shell displacement
x	axial co-ordinate
Y_n	Bessel function of order n
ϕ	deformation potential of the fluid
$\tilde{\phi}$	velocity potential of the fluid
ν	Poisson ratio
ρ_{F_i}	mass density of the internal fluid
ρ_{F_o}	mass density of the external fluid
ρ_T	mass density of shell or tube
θ	angular co-ordinate
ω	radian frequency